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Susy CP^{N-1} model and surfaces in $\mathbb{R}^{N^2-1}$

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Abstract
We describe surfaces in R^{N^2-1} generated by the holomorphic solutions of the supersymmetric CP^{N-1} model. We show that these surfaces are described by the fundamental projector constructed out of the solutions of this model and that in the CP^1 case the corresponding surface is a sphere. Although the coordinates of the sphere are superfields the sphere's curvature is constant. We show that for $N > 2$ the corresponding surfaces can also be constructed from a similar projector.

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1. Introduction

The subject of Weierstrass representations of surfaces immersed in multidimensional spaces was introduced few years ago by Konopelchenko *et al* [1, 2]. This has generated interest [3, 4] in looking at the properties of these surfaces and relating them to the solutions of the CP^{N-1} model. Recently one of us (WJZ), together with Grundland [5] presented a general procedure for the construction of such surfaces from the harmonic CP^{N-1} maps. This approach involved writing the equation for the harmonic map as a conservation law and then observing that in this construction a special operator played a key role. This operator, related to the fundamental projector of the harmonic map, was then used in the construction of the surface.

The CP^{N-1} model has been supersymmetrized [6] thus giving us supersymmetric harmonic maps. The question then arises: (1) what surfaces do these supersymmetric maps correspond to, and (2) what properties do they have? This is the problem that is studied in this paper.

In the next section we briefly review the supersymmetric CP^{N-1} harmonic maps (using the formalism as given in [7]). We then construct the operators which are the supersymmetric generalization of the operators of the purely bosonic maps. We also construct the Weierstrass surfaces and show that, as in the purely bosonic model, the surfaces are described by the projector of the harmonic map. This allows us to show that in the CP^1 case, as in the corresponding bosonic case, the resultant surface is a sphere. In the final sections of the paper we discuss its properties and present a short discussion of the corresponding surfaces for $N > 2$.

2. Formalism

2.1. The supersymmetric CP^{N-1} model

We are interested here in the supersymmetric (SUSY) CP^{N-1} model which is constructed on the two-dimensional superspace $(x, y, \theta_1, \theta_2)$ where the anticommuting quantities θ_1 and θ_2 denote two components of a majorana spinor θ and can be thought of as being real. For our considerations, a better choice of coordinates will be the complexified superspace $(x_+, x_-, \theta_+, \theta_-)$ where

$$x_\pm = x \pm \mathrm{i}y, \qquad \theta_\pm = \theta_1 \pm \mathrm{i}\theta_2.$$

We consider in particular a complex bosonic superfield which is a N-column vector defined as

$$\Phi(x_+, x_-, \theta_+, \theta_-) = z(x_+, x_-) + \mathrm{i}\theta_+\chi_+(x_+, x_-) + \mathrm{i}\theta_-\chi_-(x_+, x_-) - \tfrac{1}{2}\theta_+\theta_- F(x_+, x_-), \qquad (2.1)$$

where z, F are N-component bosonic fields and χ_+, χ_- are N-component fermionic fields. Since the fermionic fields χ_+, χ_- anticommute with each other and with θ_+, θ_-, the Hermitian conjugate of the superfield Φ is given by

$$\Phi^\dagger(x_+, x_-, \theta_+, \theta_-) = z^\dagger(x_+, x_-) + \mathrm{i}\theta_-\chi_+^\dagger(x_+, x_-) + \mathrm{i}\theta_+\chi_-^\dagger(x_+, x_-) - \tfrac{1}{2}\theta_+\theta_- F^\dagger(x_+, x_-).$$

In the SUSY CP^{N-1} model, Φ satisfies $\Phi^\dagger\Phi = 1$. In terms of z, χ_+, χ_- and F, this condition reads

$$z^\dagger z = 1, \qquad \chi_\pm^\dagger z + z^\dagger\chi_\pm = 0, \qquad F^\dagger z + z^\dagger F = 2\big(\chi_-^\dagger\chi_- - \chi_+^\dagger\chi_+\big). \quad (2.2)$$

The usual derivatives $\partial_\pm = \frac{1}{2}(\partial_x \pm \mathrm{i}\partial_y)$ are generalized to superderivatives so that we get

$$\check{\partial}_\pm = -\mathrm{i}\partial_{\theta_\pm} + \theta_\pm\partial_\pm. \qquad (2.3)$$

They are fermionic and satisfy anticommuting properties that we have to take into account in the calculations. For example, the following relations will be useful later:

(1) If Φ is a bosonic superfield, we have

$$(\check{\partial}_\pm\Phi)^\dagger = \check{\partial}_\mp\Phi^\dagger, \qquad (\check{\partial}_+\check{\partial}_-\Phi)^\dagger = \check{\partial}_+\check{\partial}_-\Phi^\dagger. \qquad (2.4)$$

(2) If Ψ is a fermionic superfield, we have

$$(\check{\partial}_\pm\Psi)^\dagger = -\check{\partial}_\mp\Psi^\dagger, \qquad (\check{\partial}_+\check{\partial}_-\Psi)^\dagger = \check{\partial}_+\check{\partial}_-\Psi^\dagger. \qquad (2.5)$$

(3) In general, we have

$$\check{\partial}_\pm\check{\partial}_\pm = -\mathrm{i}\partial_\pm. \qquad (2.6)$$

Let us recall that we are considering SUSY models. This means that the corresponding Lagrangian density and equations of motion must be expressed in terms of the superfields Φ, $\Phi^\dagger$ and the associated supercovariant derivatives. A definition of these supercovariant derivatives has thus to be given. Let us note that they will be dependant on the superfields Φ and $\Phi^\dagger$ and will be defined as acting on bosonic as well as fermionic superfields. We get

$$\check{D}_\pm\Lambda = \check{\partial}_\pm\Lambda - \Lambda A_\pm, \qquad A_\pm = \Phi^\dagger\check{\partial}_\pm\Phi,$$

where Λ is an arbitrary homogeneous (bosonic or fermionic) superfield. In our SUSY CP^{N-1} model, the quantities $A_\pm$ are scalar fermionic superfields. In particular, we have

$$\check{D}_\pm \Phi = (\mathbb{I} - \mathbb{P})\check{\partial}_\pm \Phi,$$

where $\mathbb{I}$ is the identity operator and $\mathbb{P} = \Phi\Phi^\dagger$ is a projection operator. We also have

$$(\check{D}_\pm \Phi)^\dagger = \check{\partial}_\mp \Phi^\dagger (\mathbb{I} - \mathbb{P}).$$

We can now write both the Lagrangian density and the equations of motion of our model as

$$\mathcal{L} = 2(|\check{D}_+\Phi|^2 - |\check{D}_-\Phi|^2) \tag{2.7}$$

and

$$\check{D}_+\check{D}_-\Phi + |\check{D}_-\Phi|^2\Phi = 0. \tag{2.8}$$

Similarly to the case of the non-SUSY CP^{N-1} model, we can introduce the following spectral equations ($\lambda \in \mathbb{R}$):

$$\check{\partial}_+\Lambda = \frac{2}{1+\lambda}\mathbb{K}^\dagger\Lambda, \qquad \check{\partial}_-\Lambda = \frac{2}{1-\lambda}\mathbb{K}\Lambda,$$

where

$$\mathbb{K} = [\check{\partial}_-\mathbb{P}, \mathbb{P}], \qquad \mathbb{K}^\dagger = [\check{\partial}_+\mathbb{P}, \mathbb{P}]. \tag{2.9}$$

So the equation of motion (2.8) is a compatibility condition for these spectral equations that could be written as a superconservation law:

$$\check{\partial}_+\mathbb{K} + \check{\partial}_-\mathbb{K}^\dagger = 0.$$

Let us now show that $\mathbb{K} = \mathbb{M} + \mathbb{L}$ is in fact a linear combination of two distinct conserved quantities. Indeed, since we have $\Phi^\dagger\Phi = 1$, we can set

$$\Phi = |w|^{-1}w \tag{2.10}$$

and

$$\mathbb{P} = \Phi\Phi^\dagger = |w|^{-2}ww^\dagger.$$

Let us recall that we thus get

$$\operatorname{tr}\mathbb{P} = 1, \qquad \text{and} \qquad \det\mathbb{P} = 0. \tag{2.11}$$

Now $\mathbb{K}$ in (2.9) takes the form

$$\mathbb{K} = [\check{\partial}_-\mathbb{P}, \mathbb{P}] = |w|^{-2}(\check{\partial}_-ww^\dagger - w\check{\partial}_-w^\dagger) + |w|^{-4}(\check{\partial}_-w^\dagger w - w^\dagger\check{\partial}_-w)ww^\dagger. \tag{2.12}$$

Setting

$$\mathbb{M} = (\mathbb{I} - \mathbb{P})\frac{\check{\partial}_-ww^\dagger}{|w|^2}, \qquad \mathbb{L} = -\frac{w\check{\partial}_-w^\dagger}{|w|^2}(\mathbb{I} - \mathbb{P}), \tag{2.13}$$

we easily get $\mathbb{K} = \mathbb{M} + \mathbb{L}$. Since we also have

$$\mathbb{M} - \mathbb{L} = \check{\partial}_-\mathbb{P},$$

$\mathbb{L}$ and $\mathbb{M}$ are conserved.

Incidentally the equations (2.8) when written in terms of w take the form:

$$\check{\partial}_+\check{\partial}_-w - \check{\partial}_+w\frac{(w^\dagger\check{\partial}_-w)}{|w|^2} - \frac{(w^\dagger\check{\partial}_+w)}{|w|^2}\check{\partial}_-w - \frac{(w^\dagger\check{\partial}_+\check{\partial}_-w)}{|w|^2}w + 2w\frac{(w^\dagger\check{\partial}_+w)(w^\dagger\check{\partial}_-w)}{|w|^4} = 0. \tag{2.14}$$

2.2. Special solutions of the equations of motion

Let us now take Φ as in (2.10) where we assume that $w = w(x_+, \theta_+)$. Since, we have in this case

$$\check{\eth}_-\Phi = (\check{\eth}_-|w|^{-1})w + |w|^{-1}\check{\eth}_- w = (\check{\eth}_-|w|^{-1})w$$

and

$$\mathbb{P}\check{\eth}_-\Phi = \check{\eth}_-\Phi,$$

we get

$$\check{D}_-\Phi = 0. \tag{2.15}$$

Thus we see that

$$w = w(x_+, \theta_+)$$

solves the equation of motion (2.14). In analogy with the purely bosonic case we shall call such a solution 'holomorphic'.

In this case we also have $\mathbb{M} = 0$ and

$$\mathbb{K} = \mathbb{L} = -\check{\eth}_-\mathbb{P} = |w|^{-2}(-w\partial_- w^\dagger) + |w|^{-4}(\partial_- w^\dagger w)ww^\dagger. \tag{2.16}$$

Let us now define the bosonic quantity $\mathbf{L} = -\mathrm{i}\check{\eth}_-\mathbb{L}$ and the Hermitian conjugate which, from (2.5), is given by $\mathbf{L}^\dagger = \mathrm{i}(\check{\eth}_-\mathbb{L})^\dagger = -\mathrm{i}\check{\eth}_+\mathbb{L}^\dagger$. From (2.16), we get

$$\mathbf{L} = \partial_-\mathbb{P}, \qquad \mathbf{L}^\dagger = \partial_+\mathbb{P},$$

so that $\mathbf{L}$ is conserved in the usual sense, i.e.

$$\partial_+\mathbf{L} + \partial_-\mathbf{L}^\dagger = 0.$$

Similarly as in the non-SUSY case, we can construct

$$\mathbf{X} = \int_\gamma \mathbf{L}\,\mathrm{d}x_- + \int_\gamma \mathbf{L}^\dagger\,\mathrm{d}x_+, \tag{2.17}$$

which is independent on the contour of integration and we see that

$$\mathbf{X} = \mathbb{P}. \tag{2.18}$$

As our projector $\mathbb{P}$, seen as a $N \times N$ matrix, is Hermitian and satisfies (2.11), it can be characterized by $(N^2 - 1)$ real entries subject to a nonlinear constraint ($\det \mathbb{P} = 0$). In fact, these entries could serve to construct a real vector which will describe the surface we want to characterize.

3. The CP^1 case

3.1. Explicit form of $\mathbf{X} = \mathbb{P}$

Now we look at the case of CP^1. In this case our bosonic superfields Φ and w have only two components. Thus the projector $\mathbb{P}$ is a 2×2 matrix which can be written as

$$\mathbb{P} = \begin{pmatrix} \mathbb{P}_{11} & \mathbb{P}_{12} \\ \mathbb{P}_{21} & \mathbb{P}_{22} \end{pmatrix} = \frac{1}{2}(\mathbb{I} + X_i\sigma_i), \tag{3.1}$$

where

$$X_1 = \mathbb{P}_{12} + \mathbb{P}_{21}, \qquad X_2 = \mathrm{i}(\mathbb{P}_{12} - \mathbb{P}_{21}), \qquad X_3 = \mathbb{P}_{11} - \mathbb{P}_{22}.$$

Then using (2.11) we easily get

$$X_1^2 + X_2^2 + X_3^2 = 1. \tag{3.2}$$

This suggests that we take the three-component vector describing our surface as $X = \{X_1, X_2, X_3\}$ which is the surface of a sphere of radius 1.

To get the explicit form of X_i we can proceed as follows.

First, using the overall gauge freedom, we can choose

$$w = \begin{pmatrix} 1 \\ W \end{pmatrix} \tag{3.3}$$

and so we see that effectively we are dealing with a bosonic superfunction W. Of course now $\mathbb{P}$ is given by

$$\mathbb{P} = \frac{1}{1+|W|^2} \begin{pmatrix} 1 & W^\dagger \\ W & |W|^2 \end{pmatrix}$$

and the components of the vector X are given by

$$X_1 = \frac{W + W^\dagger}{1+|W|^2}, \qquad X_2 = \frac{\mathrm{i}(W^\dagger - W)}{1+|W|^2}, \qquad X_3 = \frac{1-|W|^2}{1+|W|^2}. \tag{3.4}$$

However, these fields are just the fields of the alternative (S^2) description of the CP^1 model. The relation between them is given by

$$X_i = \Phi^\dagger \sigma_i \Phi.$$

Thus the situation is the same as in the purely bosonic case. In that case we also knew that for holomorphic solutions of the CP^1 model the generated surface corresponded to a sphere.

Our result showing that this surface is described by the projector $\mathbb{P}$, and then the surface vector X which is constructed from this projector, in fact, corresponds to the alternative formulation of the model, is not altered by the supersymmetrization of the model.

For the solutions of the equations of motion (2.8) we have

$$W = f + \mathrm{i}\theta_+ g, \tag{3.5}$$

where f and g are, respectively, bosonic and fermionic functions of x_+.

Putting all the expressions in (3.4) we see that the explicit form of the vector X is given by

$$\begin{aligned}
X_1 &= \frac{(f+\bar f)}{1+|f|^2} + \mathrm{i}\theta_- \frac{\bar g(1-f^2)}{(1+|f|^2)^2} + \mathrm{i}\theta_+ \frac{g(1-\bar f^2)}{(1+|f|^2)^2} + 2\theta_+\theta_- \frac{\bar g g(f+\bar f)}{(1+|f|^2)^3}, \\
X_2 &= \mathrm{i}\frac{(\bar f - f)}{1+|f|^2} - \theta_- \frac{\bar g(1+f^2)}{(1+|f|^2)^2} - \theta_+ \frac{g(\bar f^2+1)}{(1+|f|^2)^2} + 2\mathrm{i}\theta_+\theta_- \frac{\bar g g(\bar f - f)}{(1+|f|^2)^3}, \\
X_3 &= \frac{(1-|f|^2)}{1+|f|^2} - 2\mathrm{i}\theta_- \frac{\bar g f}{(1+|f|^2)^2} - 2\mathrm{i}\theta_+ \frac{g\bar f}{(1+|f|^2)^2} + 2\theta_+\theta_- \frac{\bar g g(1-|f|^2)}{(1+|f|^2)^3}.
\end{aligned}$$

We note that although the components of X satisfy (3.2) they are, in fact, superfields—i.e., they have fermionic parts.

3.2. Metric

Next we look at the metric induced on the surface. We introduce the metric by putting

$$g_{ij} = \partial_i X_k \partial_j X_k,$$

where the sum goes over all the components of X.

However, it is more convenient to change variable to the holomorphic basis and so introduce $g_{\pm\pm}$, where the indices $+(-)$ denote the x_+ (x_-) components of the metric. Then, as we shall see below, only the $g_{+-} = g_{-+}$ components are nonzero.

Note that as our vector X is constructed from the components of $\mathbb{P}$ we have

$$g_{\pm\pm} = \operatorname{tr} \partial_{\pm}\mathbb{P}\partial_{\pm}\mathbb{P}. \tag{3.6}$$

Then as

$$\partial_+\mathbb{P} = -(\mathbb{I} - \mathbb{P})\frac{\partial_+ w w^\dagger}{|w|^2}$$

we see that

$$\partial_+\mathbb{P}\partial_+\mathbb{P} = 0,$$

and so we see that $g_{++} = 0$. Of course g_{--} also vanishes as it is given by $g_{--} = \bar{g}_{++}$.

However g_{+-} is nonzero. To calculate it we note that it is given by

$$\frac{\partial_+ W \partial_- \bar{W}}{[1 + |W|^2]^2}. \tag{3.7}$$

Note that this expression, superficially, is similar to the energy density. It would have been exactly that if the derivatives had been of type $\check{\partial}$ instead of type ∂. As W is a superfield g_{+-} is a superfield too. So what are its components?

Clearly, the bosonic part, which comes from putting $\theta_\pm = 0$ in (3.7) is given by

$$\frac{\partial_+ f \partial_- \bar{f}}{[1 + |f|^2]^2}.$$

It is the bosonic energy density, i.e. the term that we get in a nonsupersymmetric version of the problem. Calculating the other parts of g_{+-} we obtain the complete result as

$$g_{+-} = \frac{\partial_+ f \partial_- \bar{f}}{[1 + |f|^2]^2} + \mathrm{i}\theta_+\partial_+\left(\frac{g\partial_-\bar{f}}{[1 + |f|^2]^2}\right) + \mathrm{i}\theta_-\partial_-\left(\frac{\bar{g}\partial_+ f}{[1 + |f|^2]^2}\right) - \theta_+\theta_-\partial_-\partial_+\left(\frac{g\bar{g}}{[1 + |f|^2]^2}\right). \tag{3.8}$$

Hence we see that the metric does have fermionic corrections but, as they are total derivatives, they average to zero (i.e., vanish after integration over x_+ and x_-).

3.3. Curvature

Now, we calculate the curvature of our metric (3.8). As it has only the g_{+-} component, the curvature is given by

$$K = -2\frac{1}{F}\partial_+\partial_- \ln F, \tag{3.9}$$

where $F = \frac{1}{2}g_{+-}$.

To perform the calculation we note that

$$\ln\left(\frac{\partial_+ W \partial_- \bar{W}}{[1 + |W|^2]^2}\right) = \ln(\partial_+ W)) + \ln(\partial_- \bar{W}) - 2\ln([1 + |W|^2]). \tag{3.10}$$

However as $W = W(x_+, \theta_+)$ only the first two terms in (3.10) vanish when one applies to them $\partial_+\partial_-$ and so we get

$$K = -\frac{[1 + |W|^2]^2}{\partial_+ W \partial_- \bar{W}}(-2)\partial_+\partial_-(\ln[1 + |W|^2]) = 2. \tag{3.11}$$

Thus the curvature is purely bosonic and, as expected, is 2.

This is not unexpected as our surface is a surface of a sphere. However, it is interesting that although the coordinates of this surface are superfields and the induced metric is also described by a superfield all the fermionic effects cancel and the curvature is just $K = 2$. Hence the fermionic modification does not alter the curvature of the surface.

4. Weierstrass system for CP^1

Let us recall the regular Weierstrass problem for the nonsupersymmetric CP^1 system. In this case one considers two complex functions ψ, ϕ of x_+ and x_- which satisfy the equations

$$\partial_+\psi = (|\psi|^2 + |\phi|^2)\phi, \qquad \partial_-\phi = -(|\psi|^2 + |\phi|^2)\psi. \tag{4.1}$$

Then to find a solution of these two equations one can put

$$V = \frac{\psi}{\bar{\phi}}$$

and eliminate ψ. Then one rewrites (4.1) as

$$\partial_+V = \phi^2(1 + |V|^2)^2, \qquad \partial_-\phi^2 = -2|\phi|^4V(1 + |V|^2).$$

Thus

$$\phi^2 = \frac{\partial_+V}{(1 + |V|^2)^2} \tag{4.2}$$

and we see that V satisfies

$$\partial_-\partial_+V = 2\frac{\bar{V}\partial_+V\partial_-V}{1 + |V|^2},$$

i.e. the equation of the CP^1 model.

What is the supersymmetric version of this problem? As we know, in the supersymmetric case, V becomes W as in (3.3). Its equation of motion can be deduced easily from (3.3) and (2.14) and it is

$$\check{\eth}_+\check{\eth}_-W = 2\bar{W}\frac{\check{\eth}_+W\check{\eth}_-W}{1 + |W|^2}. \tag{4.3}$$

Having W for V, we take Z^2 as the supersymmetric analogue of ϕ^2 defined in (4.2). We require that W and Z^2 satisfy

$$\check{\eth}_+W = (1 + |W|^2)^2Z^2, \qquad \check{\eth}_+Z^2 = -2WZ^2\bar{Z}^2(1 + |W|^2).$$

Note that W is bosonic while Z^2 is fermionic. We have

$$Z^2 = \frac{\check{\eth}_+W}{(1 + |W|^2)^2} \tag{4.4}$$

and, as is easy to check, W solves equation (4.3).

Can one take the nonsupersymmetric limit of this problem? This is difficult as Z^2 is fermionic. However, we can put

$$\phi^2 = \check{\eth}_+Z^2.$$

Then, as

$$\check{\eth}_+\bar{W} = 0$$

we see that

$$\check{\eth}_+Z^2 = \frac{\check{\eth}_+\check{\eth}_+W}{[1 + |W|^2]^2}. \tag{4.5}$$

Note that due to (2.6) we see that (up to an overall factor $-\mathrm{i}$) this is the correct expression for ϕ^2 after we have set all $\theta_\pm = 0$.

5. Generalization to CP^{N-1}

Some of our results generalize easily to the CP^{N-1} case. This is the case in particular with the projector $\mathbb{P}$ which gives us a surface in R^{N^2-1} for the CP^{N-1} model.

So our surface is defined in terms of $\mathbb{P}$. How should we then define our vector X? A little thought shows that, as in the nonsupersymmetric case, we should take X in such a form that an analogue of (3.1) holds, i.e. $\partial_- X_i \partial_+ X_i$ is proportional to tr $\partial_-\mathbb{P}\partial_+\mathbb{P}$ as then $g_{++} = g_{--} = 0$.

This requires that we take off-diagonal entries of the matrix $\mathbb{P}$, say $\mathbb{P}_{ij}$ and form from them the $N(N-1)$ components of X. To do this we take $\mathbb{P}_{ij} + \mathbb{P}_{ji}$ and $\mathrm{i}(\mathbb{P}_{ij} - \mathbb{P}_{ji})$. For the diagonal entries we have some choice. We want the remaining $(N-1)$ vector components of X, called X_i to be such that

$$\sum_{i=1}^{N-1} \partial_+ X_i \partial_- X_i = 2\sum_{i=0}^{N} \partial_+ \mathbb{P}_{ii} \partial_- \mathbb{P}_{ii}.$$

In the $\mathbb{C}P^1$ case this tells us that we should take the component $\mathbb{P}_{11} - \mathbb{P}_{22}$. For larger N we have more choices; thus for CP^2 we can take (this choice is based on Gell Mann's $SU(3)$ λ matrices)

$$X_1 = \mathbb{P}_{11} - \mathbb{P}_{22}, \qquad X_2 = \sqrt{3}(\mathbb{P}_{11} + \mathbb{P}_{22}),$$

or we could make another choice. In general, for CP^2 we could take

$$\mathbb{P}_{11} = \tfrac{1}{3} + aX_1 + bX_2, \qquad \mathbb{P}_{22} = \tfrac{1}{3} + cX_1 + dX_2.$$

Then we choose a, b, c and d so that

$$\partial_+ X_1 \partial_- X_1 + \partial_+ X_2 \partial_- X_2$$

give the same expression as

$$\partial_+ \mathbb{P}_{11} \partial_- \mathbb{P}_{11} + \partial_+ \mathbb{P}_{22} \partial_- \mathbb{P}_{22} + \partial_+ \mathbb{P}_{33} \partial_- \mathbb{P}_{33}$$

in which we can eliminate $\mathbb{P}_{33}$ using $\mathbb{P}_{33} = 1 - \mathbb{P}_{11} - \mathbb{P}_{22}$.

This guarantees that only g_{+-} is nonzero. A simple calculation shows that we have a one-parameter family of solutions

$$a = \tfrac{2}{\sqrt{3}}\cos\alpha, \qquad b = \tfrac{2}{\sqrt{3}}\sin\alpha,$$
$$c = \mp\sin\alpha - \tfrac{1}{\sqrt{3}}\cos\alpha, \qquad d = -\tfrac{1}{\sqrt{3}}\sin\alpha \pm \cos\alpha.$$

For $N > 2$ the solutions are even more nonunique.

Note also that with all these choices we always have

$$g_{+-} = \mathrm{tr}(\partial_+\mathbb{P}\partial_-\mathbb{P}). \tag{5.1}$$

Moreover, the other components of the metric vanish. Thus the metric has a nontrivial dependence on the fermionic degrees of freedom. A simple calculation shows that we can rewrite (5.1) as

$$g_{+-} = (\partial_+\Phi^\dagger(\mathbb{I} - \mathbb{P})\partial_-\Phi) + (\partial_-\Phi^\dagger(\mathbb{I} - \mathbb{P})\partial_+\Phi).$$

This is closely related to the energy density of the original map—in the nonsupersymmetric case it is proportional to this density; this is not the case here as (5.1) involves ∂ derivatives and not $\check{\partial}$!

It is easy to see that the fermionic contributions to both the metric and the curvature do not cancel. We have looked at these corrections in the CP^2 case. Then the vector w has three components which can be taken in the form

$$w = \begin{pmatrix} 1 \\ W_1 \\ W_2 \end{pmatrix}. \tag{5.2}$$

The detailed calculations then show that g_{+-} is again given by the same expression as the energy density of the nonsupersymmetric model with, however, superfields in place of bosonic fields. Thus we have

$$g_{+-} = \frac{|\partial_+ W_1|^2 + |\partial_+ W_2|^2 + |W_2\partial_+ W_1 - W_1\partial_+ W_2|^2}{[1 + |W_1|^2 + |W_2|^2]^2}. \tag{5.3}$$

We can now expand this expression in powers of θ. However, it is easy to check that, say, the θ_+ corrections involve expressions that are not total derivatives. The same is true for the calculation of the curvature. In the CP^1 case we had the nice factorization of the terms in g_{+-} leading to the fact that the derivative terms did not contribute to $\partial_+\partial_- \ln(g_{+-})$. This was essential for the cancellation of various factors leading to $K = 2$. This time the numerator in (5.3) contains three terms and it does contribute to $\partial_+\partial_- \ln(g_{+-})$. In consequence K is not very simple and the fermionic contributions to it do not cancel. We have checked this explicitly but as the obtained expression is quite complicated we do not present it here. Hence, the simple results of the CP^1 case do not hold any more; both the metric and its curvature are given by full superfields.

6. Conclusions

In this paper we have discussed the supersymmetrization of the Weierstrass problem and extended to the supersymmetric case the work of Grundland *et al* [5]. Our results have shown that with small modifications the extension has not led to results which are significantly different from the purely bosonic case. In the CP^1 case we have again obtained a sphere. Its coordinates are given by real bosonic superfields which have both bosonic and fermionic parts. However, these fermionic corrections do not play a role in the description of some of the sphere's properties; e.g., in the calculation of the curvature all the fermionic contributions cancel and, as in the purely bosonic case, we get $K = 2$. They do play a role in the metric—but as they are given by total derivatives, they cancel when we integrate over x_+ and x_-.

When taking larger N we have found that, for the holomorphic CP^{N-1} fields, the projector $\mathbb{P}$ still describes the surfaces in R^{N^2-1}. This time, however, the curvature is not constant and, furthermore, it contains fermionic corrections.

The more general solutions of the supersymmetric CP^{N-1} model, for $N > 2$, are given by fields which are neither holomorphic nor antiholomorphic. Their description is somewhat complicated due to the constraints of the model. The corresponding surfaces are expected to be more complicated. They have not been studied yet due to these constraint problems which still have to be resolved. This work is currently under consideration.

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